$$\int \sqrt{x^2+9}\,dx$$

$$t = x + \sqrt{x^2+9} \rightarrow \sqrt{x^2+9} = t - x$$

$$x^2+9 = (t-x)^2$$

$$x^2+9 = t^2+x^2-2tx$$

$$2tx = t^2-9$$

$$x = \frac{t^2-9}{2t} \qquad \frac{dx}{dt} = \frac{4t^2-(t^2-9)2}{4t^2}$$

$$\frac{dx}{dt} = \frac{2t^2+18}{4t^2} = \frac{t^2+9}{2t^2}$$

$$dx = \frac{t^2+9}{2t^2}\,dt$$

$$\sqrt{x^2+9} = t - x = t - \frac{t^2-9}{2t} = \frac{2t^2-t^2+9}{2t} = \frac{t^2+9}{2t}$$

SOSTITUENDO

$$\int \sqrt{x^2+9}\,dx = \int \frac{t^2+9}{2t} \cdot \frac{t^2+9}{2t^2}\,dt = \int \frac{t^4+18t^2+81}{4t^3}\,dt$$

SPEZZANDO LA FRAZIONE

(propr. distributiva della divisione)

$$= \frac{1}{4}\int t\,dt + \frac{9}{2}\int \frac{1}{t}\,dt + \frac{81}{4}\int t^{-3}\,dt =$$

$$= \frac{1}{4}\frac{t^2}{2} + \frac{9}{2}\ln|t| + \frac{81}{4}\frac{t^{-2}}{-2} + c =$$

$$= \frac{1}{8}t^2 + \frac{9}{2}\ln|t| - \frac{81}{8}\frac{1}{t^2} + c$$

ricordando che $t = x + \sqrt{x^2+9}$

$$= \frac{3}{8}\left(x+\sqrt{x^2+9}\right) + \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + \frac{81}{8}\,\frac{1}{\left(x+\sqrt{x^2+9}\right)^2}$$

$$+ C$$